

A time-extended Hamiltonian formalism

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Abstract

A Poisson structure on the time-extended space $I \times M$ is shown to be appropriate for a Hamiltonian formalism in which time is no more a privileged variable and no *a priori* geometry is assumed on the space M of motions. Possible geometries induced on the spatial domain M are investigated. An abstract representation space for $sl(2, R)$ algebra with a concrete physical realization by the Darboux-Halphen system is considered for demonstration. The Poisson bi-vector on $I \times M$ is shown to possess two intrinsic infinitesimal automorphisms one of which is known as the modular or curl vector field. Anchored to these two, an infinite hierarchy of automorphisms can be generated. Implications on the symmetry structure of Hamiltonian dynamical systems are discussed. As a generalization of the isomorphism between contact flows and their symplectifications, the relation between Hamiltonian flows on $I \times M$ and infinitesimal motions on M preserving a geometric structure therein is demonstrated for volume preserving diffeomorphisms in connection with three-dimensional motion of an incompressible fluid.

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(1) The idea of incorporating time into geometric constructions for dynamical systems in order to get rid of its distinguished role in parametrizing the solution curves as well as those offered by the canonical Hamiltonian formalism was mainly analysed in covariant framework [1]-[3]. Most of these approaches to time-dependent dynamical systems are based on particular choices of either a symplectic or a contact structure on the spatial domain M of the flow depending on its dimension and make use of the powerful techniques of covariant geometry.

Owing to the fact that the basic object of a Hamiltonian formulation, namely, the Poisson bracket, is associated with a bi-vector, we expect to find a complete and natural understanding of Hamiltonian geometries of time-dependent dynamical systems in a contravariant treatment. With this motivation, together with the idea that a time-dependent system becomes autonomous in one higher dimensions, we wish to present a Hamiltonian framework for dynamical systems in which time is no more a privileged variable and no *a priori* geometry is assumed on the spatial domain M . We shall utilize an appropriate algebraic representation of dynamical systems as first order differential equations and contravariant geometry of Poisson structures on the time-extended space $R \times M$.

We shall show that the Hamiltonian formalism on a time-extended setting has the property that it allows us to generate recursively its infinitesimal automorphisms. This has consequences on the symmetry structure of its Hamiltonian vector fields. Applications to three-dimensional fluid motion in Lagrangian description will enable us to obtain explicit realizations of the infinite dimensional left Lie algebra of the group of volume preserving diffeomorphisms generating the particle relabelling symmetry and the associated conservation laws.

Although, the present formalism is particularly suited for non-autonomous systems, it also gives interesting results for autonomous dynamical systems with time-dependent invariants. We shall show that the algebraic and geometric structure on the phase space of the Darboux-Halphen system are implicit in its time-dependent symmetries which are eventually what dictate us to use a time-extended formalism.

(2) A non-autonomous dynamical system is represented by a time-dependent vector field which is defined to be the map $v : I \times M \rightarrow TM$ such that for $t \in I \subset R$ and $m \in M$ we have $v(t, m) \in T_m M$. The flow of v on M is a

one-parameter family g_t of diffeomorphisms given by the solution of system of differential equations

$$\frac{dx}{dt} = \frac{dg_t(x_0)}{dt} = (v_t \circ g_t)(x_0) = v_t(x) = v(t, x), \quad g_0 = id_M \quad (1)$$

where components of x are local coordinates around m and $v(t, x)$ is the representative of v in these coordinates. Eqs.(1) are equivalent to the autonomous differential equations

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = v(t, x) \quad (2)$$

on $I \times M$ associated with the suspension

$$V(t, m) = \partial_t + v(t, m) \in I \times T_m M \quad (3)$$

of v at the point (t, m) . The idea of going up one higher dimensions can be seen to be associated with the algebraic representation of first order ordinary differential equations as a subvariety of the first jet bundle over $I \times M$, identified with $I \times TM$ and embedded into $T(I \times M)$ [4]. Each embedding corresponds to a different parametrization of trajectories of v . Throughout, we shall consider only the affine parametrization (2) with time t which is taken to be the canonical coordinate on the open interval $I \subset \mathbb{R}$.

(3) By a Hamiltonian structure on a smooth n -dimensional manifold N we shall mean a bilinear mapping

$$\{, \} : C^\infty(N) \times C^\infty(N) \rightarrow C^\infty(N) \quad (4)$$

on the space of smooth functions satisfying the conditions of skew-symmetry $\{f, g\} = -\{g, f\}$ and the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \quad (5)$$

for arbitrary $f, g, h \in C^\infty(N)$, together with an isomorphism into the Lie algebra of vector fields on N . Note that, we do not require the vector field $\{f, \}$ to be a derivation on $C^\infty(N)$. This will enable us to deal with local bracket of functions defined, for example, by contact or conformally symplectic structures.

If, on the other hand, $\{ , \}$ satisfies the Leibniz' rule, then it is a Poisson bracket and is associated with a bi-vector field $P : N \rightarrow \Lambda^2(TN) = TN \wedge TN$ on N . For $N = I \times M$ we take

$$P = B \wedge \partial_t + E \quad (6)$$

where B and E are time-dependent vector and bi-vector fields on M , respectively. The Jacobi identity (5) for $\{ , \}$ is equivalent to the vanishing of the Schouten bracket $[P, P]$ of P with itself.

The Schouten bracket is defined on the space $\Lambda(TN) = \bigoplus_{k=0}^n \Lambda^k(TN)$ of multi-vector fields over N by

$$\omega([P, Q]) = (-1)^{pq+q} i(P) di(Q)(\omega) + (-1)^p i(Q) di(P)(\omega) - d\omega(P \wedge Q) \quad (7)$$

where $P \in \Lambda^p(TN)$, $Q \in \Lambda^q(TN)$, $\omega \in \Lambda^{(p+q-1)}(T^*N)$ and $i(P)(\cdot)$ is the interior product [5]. The bracket satisfies the properties

$$[P, Q] = (-1)^{pq} [Q, P] \quad (8)$$

$$(-1)^{pr} [[P, Q], R] + (-1)^{qp} [[Q, R], P] + (-1)^{rq} [[R, P], Q] = 0 \quad (9)$$

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{pq+q} Q \wedge [P, R] \quad (10)$$

which are the (graded) skew-symmetry, the Jacobi identity and the Leibniz' rule, respectively. This turns $([,], \Lambda(TN))$ into a graded Lie algebra, called the Schouten algebra [6].

The Jacobi identity for P can now be computed in a coordinate independent way using properties (8)-(10) of the Schouten bracket.

Proposition 1 *(6) is a Poisson bi-vector field on $I \times M$ if and only if*

$$[E, E] = 2B \wedge \frac{\partial E}{\partial t}, \quad [E, B] = B \wedge \frac{\partial B}{\partial t}. \quad (11)$$

(4) The Hamiltonian form of the suspended vector field V on $I \times M$ is

$$V = \partial_t + v = P(dh) = B(h)\partial_t + E(dh) - h_{,t}B \quad (12)$$

where h is a time-dependent conserved function of v . Assuming (V, P, h) is a Hamiltonian system, we shall investigate the properties of P in connection with the given system v .

Proposition 2 *Assume (V, P, h) is a Hamiltonian system on $I \times M$. Define the time-dependent bi-vector field $Q = E + v \wedge B$ on M . Then*

- (i) Q is a one-parameter family of Poisson bi-vectors on M .*
- (ii) Q and P are compatible on $I \times M$.*
- (iii) h is a Casimir function of Q .*

The proof consists of showing (i) $[Q, Q] = 0$, (ii) $[P, Q] = 0$ by a straightforward algebra using Eqs.(8-10) and, (iii) $Q(dh) = 0$ follows from Eqs.(12). As a corollary, we see that any Hamiltonian system (V, P, h) on $I \times M$ can always be reduced to $(V, P - Q = B \wedge V, h)$ which is independent of E . The next result is the well-known linearization of the Jacobi identity by Hamiltonian vector fields.

Proposition 3 *Given a time-dependent system associated with v on M , the Poisson bi-vector (6) defining the Hamiltonian structure (12) of its suspension (3) satisfies the infinitesimal invariance conditions*

$$\frac{\partial B}{\partial t} + [v, B] = 0, \quad \frac{\partial E}{\partial t} + [v, E] = B \wedge \frac{\partial v}{\partial t} \quad (13)$$

which are equivalent to (11).

This gives a characterization of the vector field B as a time-dependent infinitesimal symmetry of v . Moreover, it follows from second of Eqs.(11) that B is an infinitesimal automorphism of P .

(5) To summarize, the construction of Hamiltonian structure of a given system amounts to solving the linear system consisting of Eqs.(13), (12) and the conservation law for h . As a whole, the contravariant picture relies on finding a time-dependent symmetry B . The bi-vector field E which is the pull-back of P to M is, on the other hand, the key object determining the induced Hamiltonian geometry on the flow domain M . For given v with an infinitesimal symmetry B , E can be solved from linear equations. The solution satisfies the non-linear equations (11) which, in turn, define all possible (if any) Hamiltonian structures on M . For example, if $[E, E] = 0$ one has a Poisson structure on M which, if non-degenerate, is dual to a symplectic structure. Another possible solution that results in an algebra of functions on M is given by

$$[E, E] = 2B \wedge E, \quad [E, B] = 0 \quad (14)$$

and is called a Jacobi structure. Eqs.(14) are the conditions for the local bracket

$$\{f, g\} = E(df \wedge dg) + fB(g) - gB(f) \quad (15)$$

of functions on M to satisfy the Jacobi identity (5). In covariant geometry this corresponds to either a contact or a conformally symplectic structure [7].

(7) We shall now prove a distinguished invariance property of time-extended Hamiltonian formalism. Associated to any bi-vector P on N with volume n -form ν , there corresponds a vector field $p_\nu \equiv D_\nu(P)$ where $D_\nu \equiv \nu^{-1} \circ d \circ \nu$ is the curl operator on $\Lambda(TN)$ introduced in Ref.[8] and [9]. If P is a Poisson bi-vector p_ν is an infinitesimal automorphism of P , called the modular vector field [10], and it is moreover, a Hamiltonian vector field. Following result is a consequence of the decomposition $N = I \times M$ adapted for our investigation of dynamical systems.

Proposition 4 *For the Poisson bi-vector (6) the brackets*

$$[\dots[[B, p_\nu], p_\nu], \dots,] \quad (16)$$

generate recursively an infinite hierarchy of automorphisms.

Proof: We consider the Jacobi identity (9) for the triple (P, B, p_ν) of multi-vector fields

$$0 = [[P, B], p_\nu] + [[B, p_\nu], P] - [[p_\nu, P], B] \quad (17)$$

where the first and the last terms vanish because B and p_ν are automorphisms. So, $[B, p_\nu]$ is also an automorphism. Using this result in eq.(17) with B replaced by $[B, p_\nu]$, we arrive at the conclusion. •

(8) The explicit recursive construction of invariances of P has immediate implications on the symmetry structure of Hamiltonian vector fields associated with P . To this end, we want to relate the automorphisms of P to the infinitesimal symmetries of Hamiltonian vector fields. Since $P(dh)$ and p_ν are automorphisms, so is their bracket

$$[p_\nu, P(dh)] = P(d(p_\nu(h))) \quad (18)$$

which is Hamiltonian as well. We observe that if $p_\nu(h)$ is a Casimir function of P , then p_ν is a time-dependent infinitesimal symmetry of v . Thus, in this

case, the infinitesimal automorphisms of P generated recursively by (16) can be carried, together with their Lie algebraic structure, over the algebra of infinitesimal symmetries of v . By construction, the symmetry algebra of v may well be infinite dimensional. However, the Casimir condition and hence the degeneracy of P brings restrictions on these symmetries. When P is non-degenerate so that it has only trivial Casimirs, we can set $p_\nu(h) = 0$.

The symmetry algebra consisting of vector fields on $I \times M$ of the form $U_k = \xi_k \partial_t + u_k$ can equivalently be represented by time-dependent vector fields on M . These are the unique characteristic forms $\hat{u}_k = u_k - \xi_k v$ of U_k 's along the given vector field v [11]. To generate them one replaces p_ν in Eq.(16) with

$$\hat{p}_\nu = D_\nu(Q) + \text{div}_\nu(V)B \quad (19)$$

where Q is the bi-vector introduced in proposition(2). It follows that if the volume n -form ν is invariant under the flow of v , the characteristic form of the curl of Poisson bi-vector P on $I \times M$ is the curl of the Poisson bi-vector Q on M which is compatible with P .

(9) The geometric framework at our disposal finds applications in qualitative analysis of geometric features of the flow space M . We presented in [12] an example due to Weinstein [13] in which M has the structure of the dual space of Lie algebras. In Weinstein's example a Poisson structure on $I \times \mathfrak{g}^*$ reduces to a parametric family of Lie-Poisson structures on duals of three dimensional Lie algebras having different topological structures for certain values of the parameter. Along the same line of inquiry, we shall now consider an example in which M has the structure of a Lie algebra.

Let M be the Lie algebra $sl(2, R)$ or, more generally, any smooth manifold having an action of it. Let the vector fields v, u, w satisfying the Lie bracket relations

$$[v, u] = -2v, \quad [w, u] = 2w, \quad [v, w] = u \quad (20)$$

be a basis of $sl(2, R)$ or fundamental vector fields of the action on M . Then, $v \wedge u$ and $w \wedge u$ define on M two incompatible Poisson structures and the pair $E = w \wedge v$, $B = u$ result in a Jacobi structure.

Defining the time-dependent vector fields $U_t = u + 2tv$ and $W_t = -w + tu + t^2v$ on M , we find that $(-v, U_t, W_t)$ constitute a parametric family of basis for $sl(2, R)$. Moreover, they satisfy the conditions

$$[\partial_t + v, U_t] = 0, \quad [\partial_t + v, W_t] = 0 \quad (21)$$

to be infinitesimal symmetries of v . Including the generator ∂_t of one-dimensional algebra of translations into the time-dependent basis, we obtain on $I \times M$ the structure of a semi-direct sum of algebras.

On the time-extended space $I \times M$ the Poisson bi-vectors $P_1 = (\partial_t + v) \wedge U_t$ and $P_2 = (\partial_t + v) \wedge W_t$ are compatible. P_1 reduces to $v \wedge u$ on M whereas the induced structure by P_2 is a one-parameter family of Jacobi structures

$$E_t = v \wedge (-w + tu), \quad B_t = u + 2tv \quad (22)$$

which includes $(w \wedge v, u)$ for $t = 0$.

(10) A concrete physical realization in coordinates of the above abstract algebraic setting is provided by the Darboux-Halphen system [14]

$$v(m) = (yz - xy - xz)\partial_x + (xz - xy - yz)\partial_y + (xy - xz - yz)\partial_z \quad (23)$$

possessing the time-dependent symmetry transformations [15]

$$(t, x^i) \mapsto \left(\frac{at + b}{ct + d}, 2c \frac{ct + d}{ad - cb} + \frac{(ct + d)^2}{ad - cb} x^i \right) \quad (24)$$

where (x, y, z) is a local coordinate system around m and $a, b, c, d \in R$ with $ad - cb \neq 0$. We refer to Ref.[16] and the references therein for the recent resurrection of this system in modern theoretical physics.

The lifts of the generators on $I \times M$ of the three-parameter family of transformations (24) along the vector $v(m)$ result precisely in the time-dependent basis $(-v(m), U_t(m), W_t(m))$ of $sl(2, R)$ at the point $m \in M$. Here, $v(m)$ is given by (23) and we find that u, w have the representatives

$$u(m) = 2(x\partial_x + y\partial_y + z\partial_z), \quad w(m) = \partial_x + \partial_y + \partial_z \quad (25)$$

in the adapted coordinate system [17]. Thus, starting from the transformations (24) and reading the above abstract algebraic construction backward, we recover the geometric and algebraic structure on flow space of the autonomous system (23).

(11) In covariant framework the construction of a symplectic structure on (even dimensional) $I \times M$ relies on finding a necessarily degenerate invariant two-form on M . In this case, the construction of proposition(4) follows from

Proposition 5 *Let ω be a time-dependent, closed two-form on M . If it is a relative invariant of v then $\partial_t + v$ is symplectic.*

Proof: The closure and invariance conditions imply that $d(i(v)(\omega) - \sigma) = 0$ for some one-form σ . By Poincaré lemma there exist a function h such that $dh = i(v)(\omega) - \sigma$. Solving its time dependence from $\varphi_{,t} = i(v)(\sigma)$ makes h into a conserved quantity for v and results in Hamiltonian form of $\partial_t + v$ with the two-form $\sigma \wedge dt + \omega$. This is closed on $I \times M$ provided the time dependence of ω is determined from $\omega_{,t} + d\sigma = 0$. •

For a three-dimensional smooth manifold M the covariant and contravariant approaches to the construction of Hamiltonian structure on $I \times M$ become identical. Namely, for $M \equiv R^3$, ω and σ can be associated with three component vector fields $\omega = \mathbf{B} \cdot d\mathbf{x} \wedge d\mathbf{x}$ and $\sigma = \mathbf{E} \cdot d\mathbf{x}$ where \mathbf{B} is divergence-free because ω is closed. Then, the equations in the proof of proposition(5) become

$$\nabla\varphi = -\mathbf{v} \times \mathbf{B} - \mathbf{E}, \quad \varphi_{,t} = \mathbf{v} \cdot \mathbf{E}, \quad \mathbf{B}_{,t} + \nabla \times \mathbf{E} = 0 \quad (26)$$

the last of which is equivalent to the symmetry condition on \mathbf{B} when \mathbf{E} is solved from the first.

(12) In hydrodynamical context [18],[19], \mathbf{B} can be recognized to be a so-called frozen-in field and the above construction enables one to obtain the symplectic structure of suspended velocity field in Lagrangian description of fluid motion, that is, the description by trajectories of v . Thus, in Lagrangian coordinates $\partial_t + v$ where v is governed by a general force field, is Hamiltonian with φ and the symplectic two-form

$$\Omega = \mathbf{B} \cdot d\mathbf{x} \wedge d\mathbf{x} - (\nabla\varphi + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{x} \wedge dt \quad (27)$$

for which $\rho_\varphi \equiv -\mathbf{B} \cdot \nabla\varphi$ is the Liouville density of the invariant symplectic volume. This formal symplectic set-up has been used to construct Lagrangian and Eulerian invariants of hydrodynamic flows as well as to elucidate the connection between them. The following results has been obtained in [20],[21].

The vector fields $U_k = (\mathcal{L}_{U_1})^k(U_0)$ where $U_0 = \rho_\varphi^{-1}B$ and

$$U_1 = -U_0(h)(\partial_t + v) + fU_0 + W_1, \quad W_1 \equiv \rho_\varphi^{-1}\nabla\varphi \times \nabla h \cdot \nabla$$

is the Hamiltonian vector field for the time-dependent function h satisfying $dh/dt = f(\varphi)$ for some function f , are infinitesimal symmetries of v .

If h is a conserved function of v , so are $-U_0(h_k)$ where $h_k \equiv -(W_1)^{k-2}(U_0(h))$. In this case, the characteristic vector fields associated with U_k 's reduce to the left-invariant vector fields $W_k \equiv (\mathcal{L}_{W_1})^{k-1}(U_0)$ which are ρ_φ -divergence-free. U_k, W_k and $U_k - W_k$ are Hamiltonian vector fields with the common Hamiltonian function h_k and the Poisson bi-vectors $P \equiv \Omega^{-1}, Q$ and $P - Q$, respectively. The Poisson bracket algebra on M of functions h_k with the bracket defined by Q is isomorphic to the algebra of vector fields W_k generating the time-dependent volume preserving diffeomorphisms (known as particle relabelling symmetries) on M .

(13) We shall now show that the formal symplectic structure (27) can also be used to study the geometry of invariants of v . In particular, we shall consider the Eulerian conservation law of helicity and its degeneration into a Lagrangian invariant in the geometric language of Hamiltonian structures on $I \times M$. We assume that $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector potential \mathbf{A} . Then the symplectic two-form (27) is exact $\Omega = -d\theta$ with $-\theta = \psi dt + \mathbf{A} \cdot d\mathbf{x}$. The canonical one-form θ is a relative invariant of v

$$\mathcal{L}_{\partial_t + u}(\theta) = d\chi, \quad \chi \equiv \psi + \varphi + \mathbf{v} \cdot \mathbf{A} \quad (28)$$

and it becomes an absolute invariant if χ is a constant. The three form $\theta \wedge d\theta$ is also a relative invariant

$$\mathcal{L}_{\partial_t + u}(\theta \wedge d\theta) = d(\chi\Omega) \quad (29)$$

and it results in (magnetic) helicity conservation of Eulerian description via the identity $d(\theta \wedge d\theta) - \Omega \wedge \Omega = 0$. When $\chi = \text{constant}$, $\theta \wedge d\theta$ becomes an absolute invariant and this turns helicity into a Lagrangian invariant, that is, conserved function of the velocity field v .

Consider now the two-forms $\Omega_\chi \equiv \chi\Omega$ for $\chi \neq \text{constant}$. Since Ω is closed they satisfy the conditions

$$d\Omega_\chi = \eta \wedge \Omega_\chi, \quad d\eta = 0, \quad (30)$$

$\eta \equiv d\log\chi$, to define a conformally symplectic structure on $I \times M$ [7]. Thus, in addition to the symplectic structure Ω , we have the family Ω_χ of conformally symplectic structures on $I \times M$ that coincide with Ω on the hypersurfaces $\chi = \text{constant}$ in $I \times M$ on which Eulerian conserved densities become Lagrangian invariants.

Denoting the bi-vector dual to Ω by P , one finds that the contravariant version of (30) is the Jacobi structure defined by the pair

$$P_\chi \equiv \frac{1}{\chi} P, \quad W_\chi \equiv -\frac{1}{\chi^2} P(d\chi) \quad (31)$$

and is conformally equivalent to P . We therefore conclude that the absolute invariance of the canonical one-form, the degeneration of Eulerian conservation law of helicity into a Lagrangian one and, the conformal equivalence of the local structure (30) to the symplectic structure (27) as well as of their contravariant versions are all the same.

(14) We have presented a conceptually straight and technically natural approach to the Hamiltonian structure of dynamical systems without any assumption on time dependence of either the systems themselves or their invariants and, on dimensionality or geometry of the spatial domain. We showed, with examples, various applications of the time-extended Hamiltonian formalism. Application to the Darboux-Halphen system indicates that this framework may be useful to obtain interesting results even for autonomous systems. We conclude with the following remarks on the time-extended Hamiltonian formalism and its applications.

- The time-extended framework provides alternative approaches to problems arising from the distinguished role of time either its use in parametrization of solution curves or those offered by the canonical Hamiltonian formalism as well as to geometric constructions involving time variable. The construction is free from any particular parametrization of solution curves on M and is suitable for the study of reparametrized systems.
- All possible Hamiltonian geometries on the flow space M , including the local Lie algebra of functions, can be obtained from the Poisson structure on $I \times M$. This is in contrast to the approaches to time-dependent systems that specify a contact or symplectic structure on M according to its dimensionality [1],[3]. Since we have $H_{DR}(R \times M) = H_{DR}(M)$ for the cohomology spaces [22], one can even think of exploiting the structure on $I \times M$ to study the global geometric properties of the flow space M .

- If the Hamiltonian structure on $I \times M$ is obtained as a symplectification of a contact structure on M , one can find an isomorphism between contact flows on M and Hamiltonian flows on $I \times M$ [7]. The present framework generalizes this correspondence with Hamiltonian flows on $I \times M$ to other infinitesimal motions on M preserving a given geometric structure. The motion of an incompressible fluid is a manifestation of this with the diffeomorphisms preserving the symplectic volume.
- The modular vector field is intrinsically associated to any Poisson bi-vector on an oriented manifold. It is an element of the algebra of infinitesimal Poisson automorphisms and measures the extent to which Hamiltonian vector fields are divergence free [10]. A Poisson bi-vector on $I \times M$ provides us with another intrinsic infinitesimal automorphism and there follows the recursive construction of infinitely many of them.
- The idea is then to realize the symmetries of the motion on M in the automorphism group of the Poisson structure on $I \times M$. This is similar to the way one obtains infinitely many symmetries from a bi-Hamiltonian structure [23],[11] or, invariants of geodesic motions from the isometries of metric tensor [24]. In all cases one sets up a realization of symmetries of infinitesimal motions (vector fields) in the invariance group of a geometric structure (tensor field(s)). Thus, the time-extended Hamiltonian formalism results in another manifestation of this philosophy of Felix Klein with the technology of Sophus Lie [25].

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